Comment on "Viscous hydrodynamics relaxation time from AdS/CFT correspondence"

Standard hydrodynamics does not satisfy causality, and the causal theory of hydrodynamics is known as "causal hydrodynamics" [1, 2]. From an effective theory point of view, restoring causality forces one to consider higher orders in expansion. This means that causal hydrodynamics require a new set of transport coefficients in addition to ordinary transport coefficients such as shear viscosity η. One such coefficient is τ_π, the relaxation time for the shear viscous stress. Reference [3] determines the coefficient of the $\mathcal{N}=4$ SYM from AdS/CFT correspondence. The purpose of this comment is to point out that the value of τ_π is 3 times larger than their result if one takes into account an additional term in the hydrodynamic equation.[1] To make our point clear, the gravity computation done by Ref. [3] itself remains valid [Eq. (5)], and the difference lies in the hydrodynamic interpretation.

Reference [3] considers a $\mathcal{N}=4$ expanding plasma. For a boost-invariant plasma, which is often considered in the study of heavy-ion collisions, the following coordinate system is useful:

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_\perp^2 \ , \tag{1}$$

where τ, y, and $x_\perp$ are proper time, rapidity, and the transverse coordinates, respectively. Basic equations in hydrodynamics are the conservation equation and the constitutive equation. For the expanding plasma, they are given by (See, *e.g.*, Ref. [4].)

$$\begin{aligned} \partial_\tau \epsilon &= -\frac{\epsilon+p}{\tau} + \frac{\Phi}{\tau} \ , & (2)\\ \tau_\pi \partial_\tau \Phi &= -\Phi + \frac{4\eta}{3\tau} \\ &\quad -\frac{1}{2}\tau_\pi \Phi \left\{ \frac{1}{\tau} + \frac{T}{\beta_2}\frac{d}{d\tau}\left(\frac{\beta_2}{T}\right)\right\} \ , & (3) \end{aligned}$$

where ϵ is the energy density, p is the pressure, $\Phi := -\tau^2 \pi^{yy}$ is the dissipative part of the energy-momentum tensor, and $\beta_2 := \tau_\pi/(2\eta)$.

The last term in Eq. (3) needs some explanation because it plays an important role in our comment. The term is not included in the original Israel-Stewart theory (nor in Ref. [3]), but it is mandatory to ensure the second law of thermodynamics. As far as we are aware, it has been first added by Muronga [4]. On the other hand, the term is normally higher orders in the deviations from local equilibrium, so it is often neglected (See, *e.g.*, [5].) This may be the reason why it is not included in the original Israel-Stewart theory. However, such a naive power counting fails for a rapidly evolving system. In fact, one can easily check that the term is the same order as the other terms using Eq. (5) below.

The $\mathcal{N}=4$ SYM is conformal, so $\epsilon = 3p$. Also,

$$\eta = C_\eta T^3, \qquad \tau_\pi = \frac{C_\tau}{T} \tag{4}$$

from dimensional grounds. The aim of Ref. [3] is to determine the constant C_τ (and C_η) from the gravity side. The gravity computation yields the following energy density:

$$\epsilon(\tau) = \frac{N_c^2}{2\pi^2}\frac{1}{\tau^{\frac{4}{3}}}\left\{1 - \frac{\sqrt{2}}{3^{\frac{3}{4}}\tau^{\frac{2}{3}}} + \frac{\sqrt{3}}{36}\frac{1+2\ln 2}{\tau^{\frac{4}{3}}} + \cdots\right\} . \tag{5}$$

We define the effective temperature T as

$$\epsilon = \frac{3}{8}\pi^2 N_c^2 T^4 \tag{6}$$

which comes from the gravity computation for the $\mathcal{N}=4$ SYM at strong coupling. Note that temperature is τ-dependent. Substituting Eq. (5) into Eq. (2) determines Φ; then, Φ determines C_τ and C_η from Eq. (3). Ignoring the last term of Eq. (3), one gets

$$\tau_\pi = \frac{1-\ln 2}{6\pi T} \ , \tag{7}$$

which is the value obtained in Ref. [3]. However, taking the last term into account, one gets

$$\tau_\pi = \frac{1-\ln 2}{2\pi T} \ , \tag{8}$$

which is 3 times larger.

This value of τ_π is also supported from a computation in a different setting [6]. In order to avoid the confusion which comes from the power counting, it is best to consider a plasma whose deviations are small from equilibrium. In this case, the naive counting does work and we obtain the exactly the same value (8).

We thank Tetsufumi Hirano for discussions.

[1] Just before we submitted this comment, a number of interesting papers appeared [7, 8, 9, 10], which study the similar problem as ours. In particular, our point was made independently in Ref. [7].

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